AN ALTERNATIVE APPROACH TO SERRE DUALITY FOR PROJECTIVE VARIETIES

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1. Introduction

The purpose of this paper is to develop some of the theory from R. Hartshorne's *Algebraic Geometry* [5, III, §7] using different techniques. A sketch of our method, which is due to Professor Hartshorne, is as follows. By Noether normalization, an integral projective scheme X of dimension n over a perfect field k admits a finite separable morphism to $\mathbb{P}^n_k$. This morphism is flat if and only if X is Cohen-Macaulay. If $f : X \to Y$ is any finite morphism of noetherian schemes, we define a functor $f^! : \mathfrak{Qco}(Y) \to \mathfrak{Qco}(X)$ and show that $\operatorname{Hom}_X(\mathcal{F}, f^!\mathcal{G}) \cong \operatorname{Hom}_Y(f_*\mathcal{F}, \mathcal{G})$ (for $\mathcal{F}$ a coherent sheaf and $\mathcal{G}$ a quasi-coherent sheaf). We then prove that if $f : X \to \mathbb{P}^n$ is a finite morphism, then $\omega_X := f^!\omega_{\mathbb{P}^n}$ is a dualizing sheaf for X, where $\omega_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$. From this the Serre duality theorem follows.

Finally, we prove that the dualizing sheaf ω_X is isomorphic to the the canonical sheaf if X is nonsingular. This part of the paper utilizes a number of facts about Dedekind domains and discrete valuation rings. In particular, the theory of the *different* comes in at a crucial moment.

The present approach has several features which make it attractive. For one thing, our approach is very "geometric" in the sense that the dualizing sheaf of X is defined directly from a finite morphism $f : X \to \mathbb{P}^n$. Also, we use a geometric characterization of the Cohen-Macaulay property as equivalent to the flatness of f. Finally, our approach to differentials provides an interesting link to some of the important concepts of algebraic number theory; the reader familiar with this theory (as found, for example, in [7]), may find the present method an intriguing alternative to the homological proof found in [5, III, Thm. 7.11].

All references in round brackets will be to [5].

2. Serre duality

Let $f : X \to Y$ be an affine morphism of schemes. Then f is quasicompact and separated; therefore, by (II, 5.8), f gives rise to a functor $f_* : \mathfrak{Qco}(X) \to \mathfrak{Qco}(Y)$. In fact, we have that for any $f \in \mathfrak{Qco}(X)$, $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module.

We want to cook up a functor $\tilde{}: \mathfrak{Qco}(Y) \cap f_*\mathcal{O}_X\text{-}\mathfrak{Mod} \to \mathfrak{Qco}(X)$ going the other way. Given any $\mathcal{G} \in \mathfrak{Qco}(Y) \cap f_*\mathcal{O}_X\text{-}\mathfrak{Mod}$, define $\tilde{\mathcal{G}} \in \mathfrak{Qco}(X)$ in the following way. Pick an open affine $\mathcal{U} = \operatorname{Spec} A \subseteq Y$. Then since f is affine, $f^{-1}\mathcal{U}$ is affine,

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say $\operatorname{Spec} B$. Since $\mathcal{G}$ is an $f_*\mathcal{O}_X$-module, $G = \Gamma(\mathcal{U}, \mathcal{G})$ is not only an A-module, but also a B-module via $f^\#$. Now on $\operatorname{Spec} B$ we can define $\tilde{\mathcal{G}}|_{\operatorname{Spec} B}$ to be $\tilde{G}$. It is easily seen that the various $\tilde{G}$'s defined on open affines in X patch to give a sheaf $\tilde{\mathcal{G}} \in \mathfrak{Qco}(X)$. [For details, see [4, §1.4].]

Observe that f_* and $\tilde{}$ define an equivalence of categories. For $\mathcal{F} \in \mathfrak{Qco}(X)$, if $\mathcal{F} = \tilde{F}$ on $\operatorname{Spec} B$ then $f_*\mathcal{F}$ is again $\tilde{F}$ on $\operatorname{Spec} A$ (with F viewed as an A-module), and so $\widetilde{f_*\mathcal{F}} \cong \mathcal{F}$ on $\operatorname{Spec} B$, hence on all of X. Likewise, if $\mathcal{G} \in \mathfrak{Qco}(Y) \cap f_*\mathcal{O}_X\text{-}\mathfrak{Mod}$, then $\mathcal{G} = \tilde{G}$ on $\operatorname{Spec} A$, and G also has a B-module structure making $\tilde{\mathcal{G}} = \tilde{G}$ on $\operatorname{Spec} B$, and so $f_*(\tilde{\mathcal{G}}) \cong \mathcal{G}$.

It will be convenient to give a quick proof here of the following fact:

Lemma 1. *Let Y be a noetherian scheme. If $\mathcal{G} = \widetilde{M} \in \mathfrak{Qco}(Y)$ and $\mathcal{F} = \widetilde{N} \in \mathfrak{Coh}(Y)$ then we have $\mathcal{H}om_X(\mathcal{F}, \mathcal{G}) \in \mathfrak{Qco}(Y)$. Furthermore, $\mathcal{H}om_X(\mathcal{F}, \mathcal{G}) = \operatorname{Hom}(N, M)\tilde{}$.*

Proof. By (II, Ex. 5.4) we have locally an exact sequence

$$\mathcal{O}_Y{}^m \longrightarrow \mathcal{O}_Y{}^n \longrightarrow \mathcal{F} \longrightarrow 0$$

to which we can apply the left exact functor $\mathcal{H}om_Y(-, \mathcal{G})$ to obtain $\mathcal{H}om_X(\mathcal{F}, \mathcal{G})$ as the kernel of a map $\mathcal{G}^n \to \mathcal{G}^m$ of quasi-coherent sheaves. But $\mathcal{H}om_X(\mathcal{F}, \mathcal{G}) = \mathcal{H}om_X(\widetilde{N}, \widetilde{M}) \in \mathfrak{Qco}(Y)$ implies, by (II, 5.5) that this sheaf is just the twiddle of its global sections $\operatorname{Hom}(N, M)$, i.e., $\mathcal{H}om_X(\widetilde{N}, \widetilde{M}) \cong \operatorname{Hom}(N, M)\tilde{}$. ■

Construction of $f^!$ Now let $f : X \to Y$ be a finite morphism of noetherian schemes. By (II, Ex. 5.5(c)), there is a corresponding functor $f_* : \mathfrak{Coh}(X) \to \mathfrak{Coh}(Y)$. It will be important for what follows for us to define a sort of adjoint to f_* which we will call $f^! : \mathfrak{Qco}(Y) \to \mathfrak{Qco}(X)$. First observe that any finite morphism is affine, so we can employ the operation $\tilde{}$ defined above. Since f is finite, $f_*\mathcal{O}_X$ is coherent, so for any $\mathcal{G} \in \mathfrak{Qco}(Y)$, the sheaf $\mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{G})$ is quasi-coherent by Lemma 1 and is also clearly an $f_*\mathcal{O}_X$-module. Thus it makes sense to define $f^!\mathcal{G} = \mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{G})\tilde{}$.

The Affine Case Before we go on, let us consider what our functors $f_*, \tilde{}$, and $f^!$ do in the case where $f : X = \operatorname{Spec} B \to Y = \operatorname{Spec} A$ corresponding to $f^\# : A \to B$. For some terminology, if A and B are any commutative rings, with $N \in A\text{-}\mathfrak{Mod}$ and $M \in (A, B)\text{-}\mathfrak{BiMod}$, we define $\underline{\mathrm{Hom}_A}(M, N)$ and $N \otimes_A M$ to be the usual Hom and tensor product but with the obvious B-module structure derived from M. Observe then that

$$- \otimes_B M \dashv \underline{\mathrm{Hom}_A}(M, -) \tag{1}$$

by [1, Thm. 7.2.20]. (The notation $F \dashv G$ denotes an adjunction between the functors F and G with F as the left adjoint and G as the right.)

The functor $\tilde{}$ is "for any A-module which is also a B-module in a manner compatible with $f^!$, consider it as a B-module." f^*, on the other hand, is the functor "consider B-modules as A-modules" . It can be described alternatively as $- \otimes_B B$ (with the A-module structure derived from B) or $\underline{\mathrm{Hom}_B}(B, -)$. Thinking of f_* in the latter interpretation, it has by (1) a left adjoint $- \otimes_A B$ which is usually called f^*.

Considering f_* in its tensor product avatar gives it a right adjoint $\underline{\mathrm{Hom}_A}(B, -)$, which we recognize as corresponding exactly to our functor $f^!$. Since f is finite, B

is finitely generated over A, so $\widetilde{B}$ is a coherent sheaf on $Y = \operatorname{Spec} A$. Again using Lemma 1 we get that indeed $f^!$ corresponds to $\underline{\mathrm{Hom}}_A(B, -)$.

Note also that for locally free sheaves $\mathcal{M}$ on Y, $f^!$ is particularly easy to compute:

Lemma 2. *Let $f : X \to Y$ be a finite morphism of noetherian schemes, and $\mathcal{M}$ a locally free sheaf on Y. Then*

$$f^!\mathcal{M} \cong f^!\mathcal{O}_Y \otimes_{\mathcal{O}_X} f^*\mathcal{M}. \tag{2}$$

Proof. Since f_* and $\tilde{}$ define an equivalence of categories, we have

$$f^!\mathcal{O}_Y \otimes_{\mathcal{O}_X} f^*\mathcal{M} \cong (f_*(f^!\mathcal{O}_Y \otimes_{\mathcal{O}_X} f^*\mathcal{M}))\tilde{},$$

but by the projection formula (II, Ex. 5.2(d))

$$(f_*(f^!\mathcal{O}_Y \otimes_{\mathcal{O}_X} f^*\mathcal{M}))\tilde{} \cong (f_*(f^!\mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \mathcal{M})\tilde{}.$$

Now by the definition of $f^!$ and the definition of the dual of an invertible sheaf,

$$(f_*(f^!\mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \mathcal{M})\tilde{} \cong ((f_*\mathcal{O}_X)^\vee \otimes_{\mathcal{O}_Y} \mathcal{M})\tilde{},$$

and finally by (II, Ex. 5.2(b)),

$$((f_*\mathcal{O}_X)^\vee \otimes_{\mathcal{O}_Y} \mathcal{M})\tilde{} \cong (\mathcal{H}\mathrm{om}_Y(f_*\mathcal{O}_X, \mathcal{M}))\tilde{} \cong f^!\mathcal{M},$$

so the formula (2) is valid. ∎

We will now prove:

Lemma 3. *For any $\mathcal{F} \in \mathfrak{Coh}(X)$ and $\mathcal{G} \in \mathfrak{Qco}(Y)$, we have*

$$f_*\mathcal{H}\mathrm{om}_X(\mathcal{F}, f^!\mathcal{G}) \cong \mathcal{H}\mathrm{om}_Y(f_*\mathcal{F}, \mathcal{G}). \tag{3}$$

Observe that by taking global sections of both sides in (3), we obtain the correspondence

$$\mathrm{Hom}_X(\mathcal{F}, f^!\mathcal{G}) \cong \mathrm{Hom}_Y(f_*\mathcal{F}, \mathcal{G}) \tag{4}$$

mentioned in the introduction.

Proof. To prove (3), we will construct a map going from the left hand side to the right, and check locally that it is an isomorphism. We will build our map up in several steps. First of all, setting $\mathcal{H} = f^!\mathcal{G}$, we have the usual map

$$f_*\mathcal{H}\mathrm{om}_X(\mathcal{F}, \mathcal{H}) \to \mathcal{H}\mathrm{om}_Y(f_*\mathcal{F}, f_*\mathcal{H}), \tag{5}$$

which on an open set $\mathcal{U} \subseteq Y$ is the map sending $\phi \in \mathrm{Hom}_{\mathcal{O}_X|_{f^{-1}\mathcal{U}}}(\mathcal{F}|_{f^{-1}\mathcal{U}}, \mathcal{H}|_{f^{-1}\mathcal{U}})$ to $\psi \in \mathrm{Hom}_{\mathcal{O}_Y|_\mathcal{U}}(f_*\mathcal{F}, f_*\mathcal{H})$, where $\psi_\mathcal{W} : \mathcal{F}(f^{-1}\mathcal{W}) \to \mathcal{H}(f^{-1}\mathcal{W})$ is defined as $\phi_{f^{-1}\mathcal{W}}$ for all open $\mathcal{W} \subseteq \mathcal{U}$.

We now see that

$$f_*f^!\mathcal{G} = f_*((\mathcal{H}\mathrm{om}_Y(f_*\mathcal{O}_X, \mathcal{G}))\tilde{}) \cong \mathcal{H}\mathrm{om}_Y(f_*\mathcal{O}_X, \mathcal{G})$$

by the equivalence of categories given by $\tilde{}$ and f_*. This gives us

$$\mathcal{H}\mathrm{om}_Y(f_*\mathcal{F}, f_*f^!\mathcal{G}) \xrightarrow{\sim} \mathcal{H}\mathrm{om}_Y(f_*\mathcal{F}, \mathcal{H}\mathrm{om}_Y(f_*\mathcal{O}_X, \mathcal{G})). \tag{6}$$

Now $f_*\mathcal{O}_X$ is an $\mathcal{O}_Y$-algebra through the map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$. By the contravariant functoriality of $\mathcal{H}\mathrm{om}$ in the first variable, this gives us an evaluation map

$$\mathcal{H}\mathrm{om}_Y(f_*\mathcal{O}_X, \mathcal{G}) \to \mathcal{H}\mathrm{om}_Y(\mathcal{O}_Y, \mathcal{G}) = \mathcal{G}.$$

By the covariant functoriality of $\mathcal{H}\text{om}_Y$ in the second variable, we get a map

$$\mathcal{H}\text{om}_Y(f_*\mathcal{F}, \mathcal{H}\text{om}_Y(f_*\mathcal{O}_X, \mathcal{G})) \to \mathcal{H}\text{om}_Y(f_*\mathcal{F}, \mathcal{G}). \tag{7}$$

Composing (5), (6) and (7) (with $\mathcal{H} = f^!\mathcal{G}$ as before), we obtained the desired natural morphism

$$f_*\mathcal{H}\text{om}_X(\mathcal{F}, f^!\mathcal{G}) \to \mathcal{H}\text{om}_Y(f_*\mathcal{F}, \mathcal{G}). \tag{8}$$

To check that this is an isomorphism locally, we can restrict our attention to the case $Y = \operatorname{Spec} A, X = \operatorname{Spec} B$ (with B an A-algebra via $f^!$), $\mathcal{F} = \tilde{M}$, with M a finitely generated B-module, and $\mathcal{G} = \tilde{N}$, with N an A-module. Then the morphism (8) corresponds to the natural map

$$\begin{array}{rcl} \psi : \text{Hom}_B(M, \underline{\text{Hom}}_{\text{A}}(B, N)) & \to & \text{Hom}_A(M \otimes_B B, N) \\ \phi & \mapsto & (m \otimes 1 \mapsto \phi(m)(1)). \end{array}$$

(Recall that $M \otimes_B B$ is just M considered as an A-module via the given map $A \to B$.) But we know this map to be an isomorphism by (1). ■

Proposition 4. *Let $f : X \to Y$ be a finite morphism of noetherian schemes. Then for all $\mathcal{F} \in \mathfrak{Coh}(X), \mathcal{G} \in \mathfrak{Qco}(Y)$, and all $i \geq 0$, there is a natural map*

$$\text{Ext}^i_X(\mathcal{F}, f^!\mathcal{G}) \to \text{Ext}^i_Y(f_*\mathcal{F}, \mathcal{G}).$$

Remark: The isomorphism of Lemma 3 is the case $i = 0$ here.

Proof. The map (5) is natural in the second variable $\mathcal{H}$, so it can be considered a natural transformation of functors. Composing with the global sections functor Γ, we get the natural transformation of functors

$$\text{Hom}_X(\mathcal{F}, -) \to \text{Hom}_Y(f_*\mathcal{F}, f_*-).$$

The second functor here is just the composition of the exact functor f_* and the zeroeth component of the universal δ-functor $\text{Ext}^i_Y(f_*\mathcal{F}, -)$. Hence $\text{Hom}_Y(f_*\mathcal{F}, f_*-)$ is the zeroeth component of the δ-functor $\text{Ext}^i_Y(f_*\mathcal{F}, f_*-)$. But since $\text{Hom}_X(\mathcal{F}, -)$ is the zeroeth component of its derived functor, the universal δ-functor $\text{Ext}^i_X(\mathcal{F}, -)$, the natural transformation above extends to a natural map of δ-functors

$$\text{Ext}^i_X(\mathcal{F}, -) \to \text{Ext}^i_Y(f_*\mathcal{F}, f_*-).$$

We can now substitute $f^!\mathcal{G}$ to get a natural map

$$\text{Ext}^i_X(\mathcal{F}, f^!\mathcal{G}) \to \text{Ext}^i_Y(f_*\mathcal{F}, f_*f^!\mathcal{G}).$$

In the proof of Lemma 3 we described the natural map $f_*f^!\mathcal{G} \cong \mathcal{H}\text{om}_Y(f_*\mathcal{O}_X, \mathcal{G}) \to \mathcal{G}$. Using the covariant functoriality of Ext^i_Y in the second variable, we get the natural map

$$\text{Ext}^i_Y(f_*\mathcal{F}, f_*f^!\mathcal{G}) \to \text{Ext}^i_Y(f_*\mathcal{F}, \mathcal{G}).$$

The composition of the last two displayed maps gives our desired map. ■

Proposition 5. *With the conditions of Proposition 4, if in addition $\mathfrak{Coh}(X)$ has enough locally frees and f is flat, then the natural map*

$$\text{Ext}^i_X(\mathcal{F}, f^!\mathcal{G}) \to \text{Ext}^i_Y(f_*\mathcal{F}, \mathcal{G})$$

is an isomorphism for all $\mathcal{F} \in \mathfrak{Coh}(X), \mathcal{G} \in \mathfrak{Qco}(Y), i \geq 0$.

Remark: If X is a quasi-projective scheme, or a nonsingular scheme, then $\mathfrak{Coh}(X)$ has enough locally frees. Also, by (III, Prop. 9.2(e)), f is flat if and only if f satisfies the condition

$$(*)\ f_*\mathcal{O}_X \text{ is a locally free } \mathcal{O}_Y\text{-module.}$$

Proof. For $\mathcal{F} = \mathcal{O}_X$ we have (III, 6.3(c))

$$\operatorname{Ext}^i_X(\mathcal{O}_X, f^!\mathcal{G}) \cong H^i(X, f^!\mathcal{G}),$$

but since f is an affine map, we see using (III, Ex. 4.1) that

$$H^i(X, f^!\mathcal{G}) \cong H^i(Y, f_*f^!\mathcal{G}) = H^i(Y, \mathcal{H}\mathrm{om}_Y(f_*\mathcal{O}_X, \mathcal{G})).$$

Now, using (II, Ex. 5.1(b)) and the fact that $f_*\mathcal{O}_X$ is locally free, we see that

$$H^i(Y, \mathcal{H}\mathrm{om}_Y(f_*\mathcal{O}_X, \mathcal{G})) \cong H^i(Y, (f_*\mathcal{O}_X)^\vee \otimes \mathcal{G}),$$

and again by (III, 6.3(c)) we have

$$H^i(Y, (f_*\mathcal{O}_X)^\vee \otimes \mathcal{G}) \cong \operatorname{Ext}^i_Y(\mathcal{O}_Y, (f_*\mathcal{O}_X)^\vee \otimes \mathcal{G}).$$

By (III, 6.7), again using condition (*), we have

$$\operatorname{Ext}^i_Y(\mathcal{O}_Y, (f_*\mathcal{O}_X)^\vee \otimes \mathcal{G}) \cong \operatorname{Ext}^i_Y(f_*\mathcal{O}_X, \mathcal{G}).$$

It is an exercise for the reader to verify that the composite of these isomorphisms is our natural map! This shows that our claim is true for $\mathcal{F} = \mathcal{O}_X$.

It is clear now that the claim is also true if $\mathcal{F}$ is a free $\mathcal{O}_X$-module of finite rank. In fact it is also true for $\mathcal{F}$ locally free of finite rank, since we are given a global map and verifying that it is an isomorphism may be done locally, which comes down to the calculation we have just done.

Since $\mathfrak{Coh}(X)$ has enough locally frees and $\mathcal{F}$ is coherent, we can find a locally free sheaf $\mathcal{E}$ of finite rank that surjects onto $\mathcal{F}$. Calling the kernel $\mathcal{R}$, we have a short exact sequence

$$0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0$$

of coherent sheaves on X. Taking f_* of this sequence, we get a short exact sequence of coherent sheaves on Y:

$$0 \to f_*\mathcal{R} \to f_*\mathcal{E} \to f_*\mathcal{F} \to 0.$$

(Note that $f_* : \mathfrak{Coh}(X) \to \mathfrak{Coh}(Y)$ is exact since it has a right adjoint $f^!$). Taking $\operatorname{Hom}_X(-, f^!\mathcal{G})$ of the first sequence and $\operatorname{Hom}_Y(-, \mathcal{G})$ of the second, and developing them into a long exact sequence of Ext's using (III, 6.4), we can use our natural maps to get a morphism of long exact sequences:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \operatorname{Hom}_X(\mathcal{F}, f^!\mathcal{G}) & \longrightarrow & \operatorname{Hom}_X(\mathcal{E}, f^!\mathcal{G}) & \longrightarrow & \operatorname{Hom}_X(\mathcal{R}, f^!\mathcal{G}) & \longrightarrow & \operatorname{Ext}^1_X(\mathcal{F}, f^!\mathcal{G}) & \longrightarrow & \operatorname{Ext}^1_X(\mathcal{E} \\
 & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \phi_4} & & \downarrow \\
0 & \longrightarrow & \operatorname{Hom}_Y(f_*\mathcal{F}, \mathcal{G}) & \longrightarrow & \operatorname{Hom}_Y(f_*\mathcal{E}, \mathcal{G}) & \longrightarrow & \operatorname{Hom}_Y(f_*\mathcal{R}, \mathcal{G}) & \longrightarrow & \operatorname{Ext}^1_Y(f_*\mathcal{F}, \mathcal{G}) & \longrightarrow & \operatorname{Ext}^1_Y(f
\end{array}$$

(Note that this long exact sequence is in fact known to be *infinitely* long and therefore it is impossible to fit on a page of finite extent such as this one.) Now by (4), ϕ_1, ϕ_2, and ϕ_3 are all isomorphisms. Since $\mathcal{E}$ is locally free of finite rank, ϕ_5 is also an isomorphism by what we have just done. Applying the "subtle 5-lemma" to the part of the diagram between ϕ_2 and ϕ_6, we get that ϕ_4 is injective. Since $\mathcal{F}$ was an arbitrary coherent sheaf, an analogous argument shows that ϕ_6 is injective. Once again applying the "subtle 5-lemma", we deduce that ϕ_4 is in fact an isomorphism (as is ϕ_6 by analogous reasoning).

We now have that ϕ_1 to ϕ_6 are all isomorphisms, and we can repeat the argument just given to show that ϕ_7 to ϕ_9 are also isomorphisms, etc. ■

Now that we have a duality theory for finite flat morphisms, we can prove the Serre Duality Theorem for projective schemes of dimension n. In doing so, we characterize those schemes which admit a finite flat morphism to $\mathbb{P}^n$.

The following lemma will allow us to apply the theory developed thus far to projective schemes.

Lemma 6 (Noether Normalization)**.** *Let X be an integral projective scheme of dimension n over a perfect field k. Then there exists a finite separable morphism $f : X \to \mathbb{P}^n$.*

PROOF. By (I, 4.8A), $K(X)/k$ is separably generated, so by (I, 4.7A) it contains a separating transcendence base, yielding an injection $K(\mathbb{P}^n) = k(Y_1, \dots, Y_n) \hookrightarrow K(X)$. By (I, 4.4), this gives a dominant morphism $f : X \to \mathbb{P}^n$, which is finite and separable, since $K(X)/K(\mathbb{P}^n)$ is. ■

Now recall the definition of a dualizing sheaf from (III, 7). If X is a proper scheme of dimension n over a field k, a *dualizing sheaf* for X is a coherent sheaf ω on X, together with a trace morphism $t : H^n(X, \omega) \to k$ such that for all coherent sheaves $\mathcal{F}$ on X, the natural pairing

$$\mathrm{Hom}(\mathcal{F}, \omega) \times H^n(X, \mathcal{F}) \to H^n(X, \omega)$$

followed by t gives an *isomorphism*

$$\mathrm{Hom}(\mathcal{F}, \omega) \xrightarrow{\sim} H^n(X, \mathcal{F})'.$$

By (III, 7.1), $X = \mathbb{P}^n_k$ has a dualizing sheaf $\omega_X = \mathcal{O}_X(-n-1)$. (This is easily established using explicit calculations of the cohomology of projective space, see (III, 5).)

Theorem 7. *Let X be a projective scheme of dimension n over a perfect field k. Then X has a dualizing sheaf.*

Remark: We know from (III, 7.2) that any two dualizing sheaves are canonically isomorphic so we will refer to "the" dualizing sheaf.

PROOF. Let $f : X \to \mathbb{P}^n$ be a finite morphism, and define

$$\omega_X := f^! \omega_{\mathbb{P}^n} = \mathcal{H}\mathrm{om}_{\mathbb{P}^n}(f_* \mathcal{O}_X, \omega_{\mathbb{P}^n})\check{}.$$

Note that since $\omega_{\mathbb{P}^n}$ is locally free, Lemma 2 applies and $\omega_X = f^! \mathcal{O}_Y \otimes_{\mathcal{O}_X} f^* \omega_{\mathbb{P}^n}$. Now ω_X, being the tensor product of two coherent sheaves, is coherent. Then we have by (4) that

$$\mathrm{Hom}_X(\mathcal{F}, \omega_X) \cong \mathrm{Hom}_{\mathbb{P}^n}(f_* \mathcal{F}, \omega_{\mathbb{P}^n}) \xrightarrow{\sim} H^n(\mathbb{P}^n, f_* \mathcal{F})' \cong H^n(X, \mathcal{F})'$$

where the last isomorphism comes from (III, Ex. 4.1). So $\mathrm{Hom}_X(\mathcal{F}, \omega_X) \xrightarrow{\sim} H^n(X, \mathcal{F})'$, and if we take $\mathcal{F} = \omega_X$, then the element $1 \in \mathrm{Hom}(\omega_X, \omega_X)$ gives a homomorphism $t : H^n(X, \omega_X) \to k$ which serves as a trace map. By functoriality, (ω_X, t) is a dualizing sheaf for X. ■

We'll need the following lemma.

Lemma 8. *Let $f : X \to Y$ be a finite morphism of noetherian projective schemes with X irreducible and Y nonsingular and irreducible. Then f is flat iff X is Cohen-Macaulay.*

Proof. [3, Exer. 18.17 (or Cor. 18.17)]. ∎

We can now state and prove the following theorem.

Theorem 9 (Serre duality)**.** *Let X be an irreducible projective scheme of dimension n over a perfect field k. Let ω_X be the dualizing sheaf on X.*

(a) *For all $i \geq 0$ and all $\mathcal{F} \in \mathfrak{Coh}(X)$, there are natural functorial maps*

$$\theta^i : \operatorname{Ext}^i(\mathcal{F}, \omega_X) \to H^{n-i}(X, \mathcal{F})'$$

such that θ^0 is the isomorphism given in the definition of a dualizing sheaf.

(b) *Moreover, the following conditions are equivalent:*
  (i) *there exists a finite flat morphism $f : X \to \mathbb{P}^n$;*
  (ii) *every finite morphism $f : X \to \mathbb{P}^n$ is flat;*
  (iii) *X is Cohen-Macaulay.*

(c) *Finally, the above three equivalent conditions all imply:*
  (iv) *The maps θ^i of (a) are isomorphisms for all $i \geq 0$ and all $\mathcal{F} \in \mathfrak{Coh}(X)$.*

Proof. The following commutative diagram defines θ^i:

$$\begin{array}{ccc}
\operatorname{Ext}^i_X(\mathcal{F}, \omega_X) & \xrightarrow{\theta^i} & H^{n-i}(X, \mathcal{F})' \\
{\scriptstyle \phi^i}\downarrow & & \uparrow{\scriptstyle \cong} \\
\operatorname{Ext}^i_{\mathbb{P}^n}(f_*\mathcal{F}, \omega_{\mathbb{P}^n}) & \xrightarrow{\cong} & H^{n-i}(\mathbb{P}^n, f_*\mathcal{F})'
\end{array}$$

where the ϕ^i are the maps coming from Proposition 4, and the bottom isomorphism comes from Serre duality for $\mathbb{P}^n$ (III, 7.1). By Proposition 5, we know that the maps ϕ^i being isomorphisms for all $i \geq 0$ and all $\mathcal{F} \in \mathfrak{Coh}(X)$ is implied by the existence of a flat morphism $f : X \to \mathbb{P}^n$. This proves (a) and (c).

(ii)⇒(i) is trivial, since Lemma 6 guarantees the existence of a finite morphism from X to $\mathbb{P}^n$. Then Lemma 8 tells us that (i)⇒(iii) and that (iii)⇒(ii). ∎

3. The dualizing sheaf

Now that we have established Serre duality, we will show that the dualizing sheaf $\omega_X := f^!\omega_{\mathbb{P}^n}$ is in fact the canonical sheaf on a nonsingular (noetherian) projective scheme X. Recall from (II, Example 8.20.1) that for $\mathbb{P}^n$, $\bigwedge^n \Omega_{\mathbb{P}^n}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-n-1)$ which we have already seen is the dualizing sheaf. We need a definition and a lemma first.

Definition 10. *If $\mathcal{T}$ is a torsion sheaf on a normal noetherian scheme X, we define the* ramification divisor *of $\mathcal{T}$ to be*

$$R = \sum_Z \operatorname{length}_{\mathcal{O}_\zeta}(\mathcal{T}_\zeta) \cdot Z,$$

where the sum ranges over all the irreducible closed subschemes Z of codimension 1 in X, and ζ denotes the generic point of Z.

Lemma 11. *Let X be a normal noetherian scheme. Suppose $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves of rank n on X and that $\mathcal{T}$ is a torsion sheaf on X with ramification divisor R, such that the sequence*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{T} \longrightarrow 0$$

is exact. Then $\bigwedge^n \mathcal{G} \cong \bigwedge^n \mathcal{F} \otimes \mathcal{L}(R)$.

Proof. We make the following definition: if M is a module of finite length over some commutative ring A, then we denote by $\chi_A(M)$ the product of the (not necessarily distinct) primes occuring in a Jordan-Hölder series for M. (For noetherian integrally closed domains, this χ_A is the canonical map from the category of finite length A-modules to its Grothendieck group, which is the group of ideals of A. See [7, I §5] for details.) Also note that since X is normal, it makes sense to talk about Weil divisors on X.

We now begin the proof of the lemma. Taking nth exterior powers, we get an exact sequence

$$0 \longrightarrow \bigwedge^n \mathcal{F} \xrightarrow{\det} \bigwedge^n \mathcal{G} \longrightarrow \mathcal{C} \longrightarrow 0.$$

Since $\bigwedge^n \mathcal{G}$ is locally free of rank 1, we can tensor by its dual to get an exact sequence

$$0 \longrightarrow \bigwedge^n \mathcal{F} \otimes (\bigwedge^n \mathcal{G})^{-1} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{C} \otimes (\bigwedge^n \mathcal{G})^{-1} \longrightarrow 0.$$

But then $\bigwedge^n \mathcal{F} \otimes (\bigwedge^n \mathcal{G})^{-1}$ equals (as subsheaves of $\mathcal{O}_X$) $\mathcal{I}_D$, the ideal sheaf of some locally principal closed subscheme of X corresponding to an (effective) Cartier divisor D, and hence $\mathcal{C} \cong (\bigwedge^n \mathcal{G}) \otimes \mathcal{O}_D$.

If we let $C(X)$ be the group of Cartier divisors on X, then there is an injection $C(X) \hookrightarrow \mathrm{Div}(X)$, where $\mathrm{Div}(X)$ is the group of Weil divisors. We claim that $D = R$ as Weil divisors. It suffices to show that $n_Z = n'_Z$ for each prime divisor Z of X, where n_Z denotes the coefficient of Z in R, and n'_Z denotes the coefficient of Z in D. Fix one such Z, and denote by ζ its generic point. Then the local ring $\mathcal{O}_\zeta$ is a discrete valuation ring, and we have the following exact sequences of $\mathcal{O}_\zeta$-modules:

$$0 \longrightarrow \mathcal{F}_\zeta \xrightarrow{u} \mathcal{G}_\zeta \longrightarrow \mathcal{T}_\zeta \longrightarrow 0$$

and

$$0 \longrightarrow \bigwedge^n \mathcal{F}_\zeta \xrightarrow{\det u} \bigwedge^n \mathcal{G}_\zeta \longrightarrow \mathcal{C}_\zeta \longrightarrow 0.$$

By the theory of modules over a principal ideal domain, we have

$$n_Z = \mathrm{length}_{\mathcal{O}_\zeta} \mathcal{T}_\zeta = \mathrm{length}_{\mathcal{O}_\zeta}(\operatorname{coker} u) = v_Z\left(\chi_{\mathcal{O}_\zeta}(\operatorname{coker} u)\right),$$

where v_Z denotes the valuation on the local ring $\mathcal{O}_\zeta$. By [7, I, §5, Lemma 3], we have $\chi_{\mathcal{O}_\zeta}(\operatorname{coker} u) = \beta \cdot \mathcal{O}_\zeta$, where $\beta \cdot \mathcal{O}_\zeta$ is the image of $\det u$. So $n_Z = v_Z(\beta) = \mathrm{length}_{\mathcal{O}_\zeta}(\mathcal{O}_D)_\zeta = n'_Z$ as claimed. This shows that $D = R$ as Weil divisors, and hence as Cartier divisors; in particular, R is a well-defined Cartier divisor! We conclude that

$$\mathcal{L}(-R) \cong \mathcal{I}_D \cong \bigwedge^n \mathcal{F} \otimes \left(\bigwedge^n \mathcal{G}\right)^{-1},$$

so $\bigwedge^n \mathcal{G} \cong \bigwedge^n \mathcal{F} \otimes \mathcal{L}(R)$. ∎

Remark: Using the terminology of (II, Ex. 6.11(b)), we have proved that the determinant $\det \mathcal{T}$ of our sheaf $\mathcal{T}$ is isomorphic to $\mathcal{L}(R)$.

We now want to prove the following:

Theorem 12. *Let $f : X \to Y$ be a finite separable morphism of degree d between integral nonsingular schemes of dimension n. Then*

$$\Omega_X^n \cong f^! \Omega_Y^n.$$

PROOF. What we will end up doing is proving in a slightly roundabout way that

$$\Omega_X^n \cong f^* \Omega_Y^n \otimes \mathcal{L}(R) \tag{9}$$

and

$$f^! \Omega_Y^n \cong f^* \Omega_Y^n \otimes \mathcal{L}(R) \tag{10}$$

for the same "ramification divisor" R.

Let us first try to prove (9). By (II, 8.11), we have an exact sequence

$$f^* \Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0.$$

Now Ω_X is locally free of rank n on X, and Ω_Y is locally free of rank n on Y, so $f^*\Omega_Y$ is locally free of rank n on X (since $f^*\mathcal{O}_Y = \mathcal{O}_X$). As f is a finite separable morphism, $K(X)/K(Y)$ is a finite separable extension so by [6, Thm. 25.3], we get $\Omega_{K(X)/K(Y)} = 0$. Hence $\Omega_{X/Y}$ is a torsion sheaf. Now letting $\mathcal{K}$ be the kernel of $f^*\Omega_Y \to \Omega_X$, we get an exact sequence at each point P of X

$$0 \to \mathcal{K}_P \to \mathcal{O}_P^n \to \mathcal{O}_P^n \to (\Omega_{X/Y})_P \to 0.$$

Tensoring this sequence with $K(X)$, which is flat over $\mathcal{O}_P$, we get

$$\dim_{K(X)} \mathcal{K}_P \otimes K(X) = -n + n + \dim_{K(X)} (\Omega_{X/Y})_P \otimes K(X) = 0,$$

so $\mathcal{K}_P$ is torsion. But $\mathcal{K}_P \subseteq \mathcal{O}_P^n$, and $\mathcal{O}_P$ is an integral domain, so we have $\mathcal{K} = 0$, i.e.

$$0 \to f^* \Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0.$$

Now taking the ramification divisor R associated to $\Omega_{X/Y}$, we have

$$\Omega_X^n \cong f^* \Omega_Y^n \otimes \mathcal{L}(R)$$

by Lemma 11, proving (9). (10) will be proved as Lemma 15. ■

In order to prove Lemma 15, some further preparation is needed. Let $f : X \to Y$ again be a finite separable morphism between nonsingular irreducible schemes X and Y. Then X is in particular Cohen-Macaulay, so $f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module (and also is an $\mathcal{O}_Y$-algebra via $f^\#$).

We want to define an $\mathcal{O}_Y$-linear "trace" map $\operatorname{Tr} : f_*\mathcal{O}_X \to \mathcal{O}_Y$ as follows. On any sufficiently small affine open $\mathcal{U} = \operatorname{Spec} A \in Y$, we have $f^{-1}\mathcal{U} = \operatorname{Spec} B$ affine, and B is a free A-module of rank $d = \deg f$. Let $\{e_1, \ldots, e_d\}$ be a basis for B over A, and for $b \in B$ write $be_i = \sum a_{ij} e_j$. Then we can set $\operatorname{Tr}(b) := \sum a_{ii}$ on B, which is independent of the chosen basis, so the local maps defined in this way glue properly to give a global map of sheaves on X. (For further discussion see [2, Chapter VI, Remark 6.5].)

By (4), the map $\operatorname{Tr} \in \operatorname{Hom}_Y(f_*\mathcal{O}_X, \mathcal{O}_Y)$ gives rise to a map $\overline{\operatorname{Tr}} \in \operatorname{Hom}_X(\mathcal{O}_X, f^!\mathcal{O}_Y)$. We will be interested in the cokernel $\mathcal{F}$ of the morphism $\overline{\operatorname{Tr}}$. We first collect some facts about Dedekind rings and then prove a proposition characterizing the sheaf $\mathcal{F}$.

Lemma 13. *Let A and B be Dedekind domains with fraction fields K and L, respectively. Suppose that L is finite separable over K, with B the integral closure of A in L. Then* *(a)* $B \otimes_A K = L$

(b) $\underline{\mathrm{Hom}}_A(B, A) \hookrightarrow \underline{\mathrm{Hom}}_K(L, K)$

(c) If I, J *are fractional ideals of* B, $I/IJ \cong B/J$.

(d) If M *is a torsion* B*-module, then* M *may be written uniquely in the form* $M = B/I_1 \oplus \cdots \oplus B/I_m$, *where* $I_1 \subseteq \cdots \subseteq I_m$ *is an ascending sequence of nontrivial ideals of* B.

PROOF. (a) Both L and $B \otimes_A K$ can be characterized as the integral closure of K in L; (b) follows from (a); (c) is an exercise for the reader; (d) see [3, Ex. 19.6(c)]. ■

Proposition 14. $\overline{\mathrm{Tr}}$ *is injective, and its cokernel* $\mathcal{F}$ *is a torsion sheaf. Moreover, the ramification divisor of* $\mathcal{F}$ *is equal to the ramification divisor of* $\Omega_{X/Y}$.

PROOF. We begin with a sequence of reductions. Since the statement of the lemma is of local nature and f is a finite morphism, we can assume that $Y = \mathrm{Spec}\, A$ and $X = \mathrm{Spec}\, B$. In fact, it suffices to prove the proposition after localizing at a height 1 prime of A; we may thus assume that $(A, \mathfrak{m})$ is a discrete valuation ring and B is a semilocal ring with maximal ideals $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ dominating $\mathfrak{m}$. As f is surjective (and hence dominant), $f^\# : A \to B$ is injective, so we may assume without loss of generality that $A \subseteq B$.

We now have $f^!\mathcal{O}_Y = \underline{\mathrm{Hom}}_{\mathrm{A}}(B, A)\tilde{}$, and by the equivalence of categories between B-modules and quasicoherent sheaves on $\mathrm{Spec}\, B$, we find that the exact sequence

$$\mathcal{O}_X \xrightarrow{\overline{\mathrm{Tr}}} f^!\mathcal{O}_Y \longrightarrow \mathcal{F} \longrightarrow 0 \tag{11}$$

corresponds to an exact sequence of B-modules

$$B \xrightarrow{\overline{\mathrm{Tr}}} B^* \longrightarrow M \longrightarrow 0, \tag{12}$$

where $B^* = \underline{\mathrm{Hom}}_{\mathrm{A}}(B, A)$.

The image of $1 \in B$ under $\overline{\mathrm{Tr}}$ is easily seen to be $\mathrm{Tr} \in B^*$. If we let K and L denote the fields of fractions of A and B respectively, then L/K is separable (since f is), so $\mathrm{Tr} \neq 0 \in B^*$ by [7, III, §3]. This shows that $\overline{\mathrm{Tr}}$ is injective.

Let K and L be the fraction fields of A and B, respectively. Note that $K = K(Y)$ and $L = K(X)$. By assumption A is Dedekind, and since B is finite over A it is integral over A. But X is nonsingular, hence normal, so B is integrally closed and thus is the integral closure of A in L. By (I, 6.3A) B is also a Dedekind ring, so applying Lemma 13 it follows that $B \otimes_A K = L$, and that we have an injection $B^* = \underline{\mathrm{Hom}}_{\mathrm{A}}(B, A) \hookrightarrow \underline{\mathrm{Hom}}_K(L, K)$. But $\underline{\mathrm{Hom}}_K(L, K)$ can be canonically identified with L via

$$\begin{array}{ccc} L & \xrightarrow{\sim} & \underline{\mathrm{Hom}}_K(L, K) \\ x & \mapsto & (l \mapsto \mathrm{Tr}_{L/K}(xl)). \end{array}$$

Under this identification, B^* corresponds to

$$B^\dagger = \{x \in L : \mathrm{Tr}_{L/K}(xB) \subseteq A\} \subseteq L.$$

Here $B^\dagger$ is a fractional ideal of B containing B; its inverse, the *different* $\mathfrak{D}_{B/A}$, is an integral ideal of B [7, III, §3]. By Lemma 13(c), we have

$$B^\dagger/B \cong B/\mathfrak{D}_{B/A}.$$

Note also that since B is a Dedekind ring, B^* is locally free of rank 1 as a B-module. Hence $f^!\mathcal{O}_Y$ is a locally free sheaf of rank 1, and we in fact see that $\mathcal{F}$ is a torsion sheaf in (11):

$$0 \longrightarrow \mathcal{O}_X \longrightarrow f^!\mathcal{O}_Y \longrightarrow \mathcal{F} \longrightarrow 0. \tag{13}$$

We now claim that $\text{length}_{\mathcal{O}_\zeta} (\Omega_{X/Y})_\zeta = \text{length}_{\mathcal{O}_\zeta} \mathcal{F}_\zeta$ for all ζ corresponding to generic points of prime divisors Z on X. So we have to show that

$$\text{length}_{B_\mathfrak{p}} M_\mathfrak{p} = \text{length}_{B_\mathfrak{p}} (\Omega_{B/A})_\mathfrak{p} \tag{14}$$

for height 1 primes $\mathfrak{p} \subseteq B$, i.e. $\mathfrak{p}_1, \dots \mathfrak{p}_r$. But this follows from [8, Theorem 4.1(1)]. (Note that since L/K is separable, $\Omega_{B/A} \otimes_B L = \Omega_{L/K} = 0$, hence $\Omega_{B/A}$ is a torsion B-module.) ∎

Remark: By Lemma 13(d), we must have

$$\Omega_{B/A} = \bigoplus_1^t B/I_i$$

for some ideals I_i in B. Then we can see from [8] that $\mathfrak{D}_{B/A}$ can be expressed as

$$\mathfrak{D}_{B/A} = \prod_1^t I_i.$$

Remark: Note that if we knew that all $k(\zeta)/k(\eta)$ were *separable* (where $\eta \in Y$ corresponds to $\mathfrak{m} \in \operatorname{Spec} A$), then we could apply [7, III, §7 Prop. 14] to obtain this result. The reader may find it amusing to try to come up with an example where the results of [7] are not sufficient.

We have all the components in place now for the proof of (10):

Lemma 15. *Letting R be the ramification divisor of $\mathcal{F}$ (or $\Omega_{X/Y}$),*

$$f^!\Omega_Y^n \cong f^*\Omega_Y^n \otimes \mathcal{L}(R).$$

Proof. Using the short exact sequence (13) we obtain, by Lemma 11,

$$f^!\mathcal{O}_Y \cong \mathcal{L}(R).$$

Tensoring this isomorphism on both sides by the sheaf $f^*\Omega_Y^n$, we see that

$$f^!\mathcal{O}_Y \otimes_{\mathcal{O}_X} f^*\Omega_Y^n \cong f^*\Omega_Y^n \otimes \mathcal{L}(R).$$

But since Ω_Y^n is locally free, by Lemma 2 we get

$$f^!\Omega_Y^n \cong f^*\Omega_Y^n \otimes \mathcal{L}(R).$$

∎

This establishes (10), so the proof of Theorem 12 is now complete.

Corollary 16. *Let X be a nonsingular projective variety of dimension n over k. Then $\omega_X \cong \Omega_X^n$.*

Proof. Let $f : X \to \mathbb{P}^n$ be a finite separable morphism whose existence is guaranteed by Lemma 6. Then $\omega_X \cong f^!\omega_{\mathbb{P}^n}$ by the proof of Theorem 7. But $\omega_{\mathbb{P}^n} \cong \Omega_{\mathbb{P}^n}^n$. The result now follows from Theorem 12. ∎

References

[1] William A. Adkins and Steven H. Weintraub: *Algebra, An Approach via Module Theory*, Graduate Texts in Mathematics 136, Springer-Verlag, 1992

[2] Allen Altman and Steven Kleiman, *Introduction to Grothendieck Duality Theory*, Lecture Notes in Mathematics 146, Springer-Verlag, 1970

[3] David Eisenbud: *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics 150, Springer-Verlag, 1995

[4] Alexander Grothendieck and Jean Dieudonné: *Eléments de Géométrie Algébrique II*, Publications Mathématiques 8, I.H.E.S., 1961

[5] Robin Hartshorne: *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer-Verlag, 1977

[6] Hideyuki Matsumura: *Commutative Ring Theory*, Cambridge studies in advanced mathematics 8, Cambridge University Press, 1986

[7] Jean-Pierre Serre: *Local Fields*, Graduate Texts in Mathematics 67, Springer-Verlag, 1979

[8] Bart de Smit: *The different and differentials of local fields with imperfect residue fields*, to appear in Edinburgh Math. Soc.

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